Stable Structures of Coalitions in Competitive and Altruistic Military Teams

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Abstract

In heterogeneous battlefield teams, the balance between team and individual objectives forms the basis for the *internal topological structure of teams*. The team structure is studied by presenting a graphical coalitional game (GCG) with Positional Advantage (PA). PA is Shapley value strengthened by the Axioms of value. The notion of team and individual objectives is studied by defining altruistic and competitive contribution made by an individual; altruistic and competitive contributions made by an agent are components of its total or marginal contribution. Moreover, the paper examines the *dynamic team effects*, by defining three online sequential decision games. These sequential decision games are based on marginal, competitive and altruistic contributions of the individuals towards team. The stable graphs under these sequential decision games are studied and found to be any connected, complete, or tree respectively.

Keywords: Graphical Coalitional Game, Positional Advantage, Marginal Contribution, Competitive Contribution, Altruistic Contribution, Sequential Decision Games, Stable Structures

1. Introduction

Battlefield teams are generally heterogeneous coalitions consisting of interacting humans, ground sensors, and unmanned airborne (UAV) or ground vehicles (UGV). In these teams relations are based on direct local interactions, yet information is conveyed along a chain of command. The balance between coalitional and individual objectives forms the basis for the *internal topological structure of coalitions*. In this paper, the coalitional structure is studied by using cooperative game theory. A novel graphical coalitional game (GCG), with Positional Advantage (PA), is presented. PA is based on Shapley value strengthened by the Axioms of value. Axioms of valve are based on its connectivity properties within the coalition. Under the Axioms of Value the GCG is convex, fair, cohesive, and fully cooperative. Three measures of the contributions of agents to a coalition are introduced: marginal contribution (MC), competitive contribution (CC), and altruistic contribution (AC).

Game theory, introduced as a formal discipline of mathematics by J. von Neumann [13], [14], is now an established mathematical discipline that deals with issues and strategies involving competitions and cooperation between several entities [15]. In the scope of mathematical game theory these entities are called players or agents [5], [15], [17], [20]. Game theory is used in

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14. ABSTRACT

In heterogeneous battlefield teams, the balance between team and individual objectives forms the basis for the internal topological structure of teams. The team structure is studied by presenting a graphical coalitional game (GCG) with Positional Advantage (PA). PA is Shapley value strengthened by the Axioms of value. The notion of team and individual objectives is studied by defining altruistic and competitive contribution made by an individual; altruistic and competitive contributions made by an agent are components of its total or marginal contribution. Moreover, the paper examines the dynamic team effects, by defining three online sequential decision games. These sequential decision games are based on marginal, competitive and altruistic contributions of the individuals towards team. The stable graphs under these sequential decision games are studied and found to be any connected, complete, or tree respectively.

15. SUBJECT TERMS

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many walks of life [7], [21] involving situations of competition and cooperation. Owing to the occurrences of such situations in engineering, game theory has found its way in engineering applications [2], [4], [5], [9], [10], [11], [16], [17], [18], [22], [23]. Game theory is primarily divided into two areas: noncooperative game theory [5], and cooperative game theory [15], [20].

Cooperative games can be divided into three classes [5]: Canonical Coalitional Games, Coalition Formation Games, and Coalitional Graph Games [4], [17]. Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time sequentially to maximize their prescribed objective functions [20]. Myerson [12] used the Shapley value [19] as the payoff to agents in a coalition defined by a graph. Jackson and Wolinsky [9] studied coalition graph games where the payoff depends on the connectivity of an agent and a cost was imposed for maintaining graph edges.

This paper defines a graphical coalitional game with novel properties. The first point of impact of the paper is based on a Value Function that is required to satisfy four formal axioms. Owing to these axioms imposed on the Value Function, it is possible to perform a rigorous study of the internal structure of teams. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the worth of an agent in a team, and when strengthened by the Axioms of Value, is called the Positional Advantage (PA). PA formalizes the notion of well-connectedness in a team. The second point of impact of the paper is to study three types of contributions of individuals within a team- the marginal (MC), competitive (CC), and altruistic (AC) [1]. The PA, which includes the formal Axioms of Value, allows to rigorously developing certain properties of these three contributions, including their dependence on the internal team structure and changes in the team structure. In this work the internal team structure is represented as a simple graph [6]. The third point of impact of the paper is the study of the *dynamic team effects* such as the formation of teams by adding agents, the destruction of teams by defection of agents, and other issues involving sequential decisions by agents to join, leave, or form teams. To study the dynamic team effects, three online sequential decision games are defined. These sequential decision games are based on the three contributions MC, CC, and AC. It is shown that the stable graphs under the objective of maximizing the MC are any connected graph. The stable graphs under the objective of maximizing the CC are the complete graph. The stable graphs under the objective of maximizing the AC are any tree.

This paper is organized as follows. In Section 2, a graphical coalitional game (GCG) having novel properties, is defined, with its axioms on the value. The Positional Advantage (PA) of an agent within a graph topology is also defined on the grounds established by Myerson [12] and Shapley [19]. Section 3 defines three types of contributions of agents in a coalition based on the communication graph structure. Marginal, competitive, and altruistic contributions are defined. Results about these three contributions of the agents based on topological graph properties are established. In Section 4, three online sequential decision games are developed on top of the GCG. The stable graph structures under each of these three games are studied. Simulation examples of online sequential decision games are presented in Section 5.

2. Positional Advantage in Graphical Coalitional Games

The notion of sequential decision games for coalition formation is detailed in Section 4; these games rest on the idea of graphical coalition game (GCG) introduced in this section. In this section a GCG satisfying certain axioms is defined. These axioms make it possible to establish the notion of Positional Advantage (PA) of agents within a coalition based on their positions within the graph. This section starts with a few essential notations of graph theory [6].

2.1 Graph Definitions

Consider a simple undirected graph $G=(V,E)$ with *V* a finite nonempty set of agents or vertices and *E* is a set of edges. Two agents are interpreted to have an edge between them if and only if they directly communicate with each other. The number of elements in *V* is called the order or size of the graph and is denoted as $|G|$, also denoted as *N*. The graph theoretic concepts including neighborhood, path, connectivity, dis-connectivity, components, cycle, tree, cut vertex, cut edge, subgraph, and induced subgraph are vital for the understanding of this paper and can be seen in [3], [6].

The following notations are used in this work. In this paper $S \subseteq G$, denotes that *S* is an induced subgraph of *G*. Whereas, an induced subgraph of *G* obtained by excluding a vertex *i* from *V* is denoted as $G\setminus\{i\}$. If an edge *e* is deleted from a graph *G* then the new graph obtained is denoted as $G-e$, and if an edge *e* is added in *G* then the new graph is denoted as $G\cup e$.

2.2 Graphical Coalitional Game

In this section a graphical coalitional game $\Gamma=(G,v)$ with transferable utility is proposed. The game is based on the communication structure *G* of the agents within a coalition. Given a graph *G*, with agents as nodes, and there exist edges between the nodes if and only if the corresponding agents directly communicate with each other. The Value Function *v* is formally defined as

$$v:2^G \to \Re \;\; with \;\; v(\phi)=v_0=0 \tag{1}$$

where 2^G is the collection of all the induced subgraphs of *G* and ϕ is the empty set. The Value Function satisfies the following Axioms of Value. In these axioms $S\in 2^G$ is an induced subgraph of *G*.

Axioms of the Value

1. If *S* is a connected component with $|S|=m$ then $v(S)=v_m \geq 0$
2. If *S* is having *k* connected components $S_i : i=1,2,...k$ with $|S_i|=m_i$ then $v(S)=\sum_i v_{m_i} : v_{m_i} \geq 0$
3. If $N \geq m > n \geq 0$ then $n.v_m \geq m.v_n$
4. If $N-1 \geq m > n \geq 0$ then . $v_{m+1}-v_m \geq v_{n+1}-v_n$

It is to be mentioned here that the Axiom 2 is according to allocation rules the coalitional graph game defined by Myerson in Section 3 of [12], where the coalitions are restricted by the underlying communication graph.

A condition on v_1 and v_2 stronger than Axiom 3 is

$$v_2 > 2v_1 \tag{2}$$

This condition is used to establish some refinements and strengthening of results.

The next definition provides a fundamental notion used in this paper.

Definition 1: Graphical Coalitional Game. Given a graph *G*, the graphical coalitional game is defined as the game $\Gamma=(G,v)$ where the Value Function *v* satisfies Axiom 1-4. ■

Remark 1: It can be established that Axiom 1 is implied by the Axiom 2 and Axiom 3 is implied by the Axiom 4. Nevertheless, Axioms 1 and 3 are retained as axioms because of the ease of their use in establishing the game properties [3]. ■

The allocation of the net value of a coalition to its individual agents is a fundamental problem in coalitional games. Allocation is the share given to each agent of the net value of the coalition efforts. It is established in [12] that for a Value Function defined in games on graphs, the Shapley value [19] provides a fair allocation. Therefore the next definition is made.

Definition 2: Positional Advantage of an Agent in the Graphical Coalitional Game. Given the GCG $\Gamma = (G, v)$, define the Positional Advantage of agent *i* as the Shapley value

$$\varphi_{G,v}(i) = \frac{1}{|G|} \sum_{S \subseteq G\setminus\{i\}} \frac{(v(S \cup \{i\}) - v(S))}{\binom{|G|-1}{|S|}} \tag{3}$$

Where the Value Function *v* satisfies the Axioms of value, $S \subseteq G \setminus \{i\}$ means *S* is an induced subgraph of the graph $G \setminus \{i\}$. Moreover, $S \cup \{i\}$ denotes an induced subgraph of *G*, containing all the agents in *S* and the agent *i*. ■

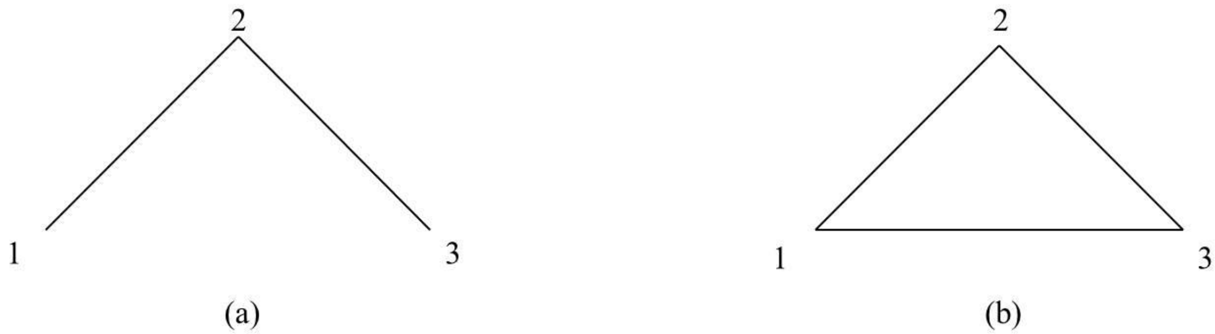

Fig. 1. Two simple graphs. (a) Example 1- Three agents in a chain (b). Example 2- Three agents in a complete graph.

The following examples explain the procedure to compute the PA of agents within a coalition and show the rationale for Axioms 1-4 of the game as laid down. They reveal the importance of PA in comparing the relative importance of agents in contributing to the communication structure of a coalition as represented by the graph *G*.

Example 1: Consider a chain of three agents $G = \{1, 2, 3\}$ as shown in Fig. 1(a). The PA of the agents is calculated by using (3). For agent 1

$$\varphi(1) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\}) - v(\{2,3\})}{1} + \tfrac{(v(\{1,2\}) - v(\{2\})) + (v(\{1,3\}) - v(\{3\}))}{2} + \tfrac{v(\{1\}) - v(\varphi)}{1}\right)$$

Using Axioms 1-4 of the game this is simplified to $\varphi(1) = \frac{1}{3}(v_3 - \frac{1}{2}v_2 + v_1)$. It can be easily seen that $\varphi(3) = \varphi(1)$. The PA of the agent 2 is given by

$$\varphi(2) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\}) - v(\{1,3\})}{1} + \tfrac{(v(\{1,2\}) - v(\{1\})) + (v(\{2,3\}) - v(\{3\}))}{2} + \tfrac{v(\{2\}) - v(\varphi)}{1}\right)$$

Using the axioms of the game this is simplified as $\varphi(2) = \frac{1}{3}(v_3 + v_2 - 2v_1)$.

Using the axioms of the game it can be readily seen that $\varphi(2) \geq \varphi(1), \varphi(3)$. This is according to the heuristics for the given communication structure, since 2 is in the middle of 1 and 3 and so logically contributes more to the communication structure of the coalition. ■

Example 2: Considering a complete graph of three agents $G = \{1, 2, 3\}$ as shown in Fig. 1(b), the PA of agent 1 is calculated by using the definition given in (3).

$$\varphi(1) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\}) - v(\{2,3\})}{1} + \tfrac{(v(\{1,2\}) - v(\{2\})) + (v(\{1,3\}) - v(\{3\}))}{2} + \tfrac{v(\{1\}) - v(\varphi)}{1}\right)$$

Using the axioms of the game this equation is simplified as $\varphi(1) = \frac{1}{3}v_3$.

It can be easily seen that both $\varphi(2)$ and $\varphi(3)$ have the same PA. This again is according to intuition, since all the three nodes are symmetrically distributed in the graph and evenly contribute to the communications within the coalition. ■

In these two examples, two small graphs were taken to demonstrate the utility of the game $\Gamma = (G, v)$ with respect to the communication structures. In these examples, the nodes that are placed more advantageously and so contribute more to the communications within a coalition have a greater PA as calculated through the payoff function (3) of the game. Moreover, the nodes which, according to the communication heuristics should have same relative importance, actually do have the same PA as calculated through the payoff function of the game.

3. Contribution of Agents within a Coalition

A graphical coalitional game is introduced in the previous section. The game assigns a value to each coalition based on communication structure represented as a graph structure. The assignment of value is based on the connectivity properties of the graph structure. All the agents involved have some allocation or share in the value assigned to the graph structure. This section defines the contribution made by an agent to the coalition and its components. The contribution and its components are further used to define sequential decision games in the next section.

Definition 5: Allocation of a Set of Agents in an Induced Subgraph. Let *A*, and *B*, be induced subgraphs of *G* such that $A \subseteq B \subseteq G$.The allocation or payoff of the agents in coalition *A* when only the coalition *B* is considered is denoted as $\mu_A(B)$ and defined as

$$\mu_A(B) = \sum_{i \in A} \varphi_B(i) \tag{4}$$ ■

3.1 Definitions of Contributions of Agents in a Coalition

The total contribution of a group *A* of agents in a coalition *G* is called the marginal contribution of *A* in *G* and written as $m_G(A)$ [1]. The marginal contribution of a group of agents is divided in two parts: one part is the contribution of the agents in the subset *A* for the sake of themselves, and the second part is the contribution of agents in *A* for the sake of the other agents in $G \setminus A$. These two parts are termed the competitive contribution $c_G(A)$ and the altruistic contribution $a_G(A)$ respectively. These contributions are formally defined next for a single agent.

Definition 6: Marginal Contribution of an Agent. The marginal contribution $m_G(i)$ of an agent *i* is defined as

$$m_G(i) = \mu_G(G) - \mu_{G \setminus \{i\}}(G \setminus \{i\}) \tag{5}$$ ■

Definition 7: Competitive Contribution of an Agent. The competitive contribution $c_G(i)$ of an agent *i* is defined as

$$c_G(i) = \mu_G(G) - \mu_{G \setminus \{i\}}(G) \tag{6}$$ ■

Definition 8: Altruistic Contribution of an Agent. The altruistic contribution $a_G(i)$ of an agent *i* is defined as

$$a_G(i) = \mu_{G \setminus \{i\}}(G) - \mu_{G \setminus \{i\}}(G \setminus \{i\}) \tag{7}$$ ■

According to these definitions

$$m_G(i) = c_G(i) + a_G(i) \tag{8}$$

Using (4) in (5), (6), (7) give

$$m_G(i) = \sum_{j \in G} \varphi_G(j) - \sum_{j \in G \setminus i} \varphi_{G \setminus i}(j) \tag{9}$$

$$c_G(i) = \sum_{j \in G} \varphi_G(j) - \sum_{j \in G \setminus i} \varphi_G(j) \tag{10}$$

$$a_G(i) = \sum_{j \in G \setminus i} \varphi_G(j) - \sum_{j \in G \setminus i} \varphi_{G \setminus i}(j) \tag{11}$$

If *G* is connected, using Axioms 1 and 2 of the graphical coalitional game and (4), equation (9) can be written as

$$m_G(i) = v_{|G|} - \sum_{j=1}^{p} v_{k_j} \ : \sum_{j=1}^{p} k_j = |G| - 1 \tag{12}$$

Here, *p* is the number of disconnected components of *G* obtained by the deletion of the agent *i* and $k_j : j = 1, 2, .., p$ are the sizes of these components.

Lemma 1 [3]: The competitive contribution of an agent in a GCG is the same as its Positional Advantage. That is

$$c_G(i) = \varphi_G(i) \tag{13}$$ ■

3.2 Dependence of Contribution of Agents on Graph Topology

Changes in marginal, competitive, and altruistic contributions are important in the online sequential decision games detailed in Section 4. Some results demonstrating the dependence of the contributions made by an agent and on graph topology and change in graph topology are presented next. Proofs are in [3].

Lemma 2: Given the GCG $\Gamma = (G, v)$ in Definition 1, in any connected graph *G* all the agents which are not cut vertices of *G* have the same marginal contribution. Moreover their marginal contribution is the minimum possible marginal contribution within the connected graph. This minimum marginal contribution is independent of the connected graph *G* and only depends upon $|G|$. ■

Remark 2: In a connected graph *G*, if there is no cut vertex then the marginal contributions of all the agents are identical and independent of the graph structure. ■

Lemma 3: In a connected graph *G* of size *N*, the maximum possible marginal contribution an agent may have is of the center point of a star ■

The next results concern the altruistic contribution.

Lemma 4: In a GCG, if an agent is isolated then its altruistic contribution is 0 ■

The next result shows that under condition (2), this result is also sufficient.

Lemma 5: In a GCG with a game having $v_2 > 2v_1$, if the altruistic contribution of an agent is 0 then it is isolated. ■

Theorem 1: In a GCG, if a new edge is formed between two connected agents *i* and *j* in a graph *G*, then the marginal contribution of its end vertices remains constant. Moreover the altruistic contributions of its end vertices change by equal non-positive values which are the negatives of the changes in the competitive contributions of its end vertices.

Proof: Let G' be the graph obtained from *G* by adding the edge $\{i, j\}$. With the help of (9) The marginal contribution of the agent *i* in the graph G' is

$$m_{G'}(i) = \sum_{j \in G'} \varphi_{G'}(j) - \sum_{j \in G' \setminus i} \varphi_{G' \setminus i}(j) \tag{14}$$

Since the new edge is formed within the same component of G, thus the first term in the right hand side of the above equation is same as the first term in the right hand side of (9). Moreover, the value of the second term is independent of any edge incident at agent *i*. Thus from (9) and (14) it is implied that $m_{G'}(i) = m_G(i)$. Rest of the result follows from (8), Lemma 1, details can be found in [3]. ■

4. Online Sequential Coalition Decision Games

Based on the marginal contribution and its components, competitive and altruistic contributions, as defined in Section 3, three online coalition sequential decision games are defined in this section on top of the graphical coalitional game $\Gamma = (G, v)$ of Definition 1, and the stable coalition graph topologies under these three games are presented. In a sequential decision game, agents take turns sequentially in time to make valid or allowed moves A background on sequential decision games can be found in Chapter 5 of [20].

4.1 Sequential Decision Games

The properties of sequential decision games depend on the allowed moves and the prescribed objective function. In the online games defined here, agents are free to make coalitions by making or breaking edges with other agents.

Allowed Moves. In these online games, at each move, an agent is selected at random. This agent is free to unilaterally break any edge incident at it or to bilaterally make an edges provided the other agent incident on the edge agrees to make it, as detailed below. In a single step, an agent is allowed either to make or break several edges.

Objective Functions. An objective function $f_G(i)$, for each agent *i* in a coalition represented by graph *G* is a real, nonnegative function. Edges are made or broken by a selected agent in order to maximize $f_G(i)$.

Based on the Allowed Moves and the Objective Function the sequential decision game is defined as follows.

Definition 9: Sequential Decision Games.

In a sequential decision game a selected agent makes or breaks edges according to the rules:

a) An agent *i* forms an edge $e = \{i, j\}$ if $f_{G\cup e}(i) > f_G(i)$ *and* $f_{G\cup e}(j) \geq f_G(j)$

b) An agent *i* breaks an edge $e = \{i, j\}$ if $f_{G-e}(i) > f_G(i)$

Based on the marginal, competitive and altruistic contributions in Section 3 the motives of agents for forming and breaking the edges are different. Taking these contributions as objective functions, three sequential decision games can be defined.

i. Game of Maximal Marginal Contribution (MMC)

In this online game the objective function $f_G(i) = m_G(i)$.

ii. Game of Maximal Competitive Contribution (MCC)

In this online game the objective function $f_G(i) = c_G(i)$.

iii. Game of Maximal Altruistic Contribution (MAC)

In this online game the objective function $f_G(i) = a_G(i)$ ■

In these three sequential decision games an agent *i* is said to have a motive to make an edge if the condition *(a)* of the game is satisfied and it is said to have a motive to break an edge if the condition *(b)* of the game is satisfied.

4.2 Stability of Graph Topologies under Sequential Decision Games

For a group of *N* agents there are $2^{N(N-1)/2}$ possible simple graphs. When agents are allowed to make valid moves, as they proceed, they may reach a graph where no agent has a motive to make any further moves. Such graphs are called stable graphs [9], [12]. The Structure of stable graphs is thus dependent on the allowed moves and the objective function of the sequential decision game.

In [12] Myerson allowed only the breakage of an edge as a valid move. Under such allowance, for the game in Definition 1 every graph is stable. In [9] the rules of making and breaking edges are nearly the same as those in Section 4.1. However, in [9] there are costs associated with making edges. The balance between the value of being connected and the cost of maintaining edges has a pivotal role in determining the stable graph structures.

Definition 10: Stable Graph. In any online sequential decision game, a graph is called stable when no agent has a motive either to make an edge or to break an edge. ■

Remark 3: It is established in [3] that the underlying game $\Gamma = (G, v)$ in Definition 1 is fair. Therefore, in the online Game of Maximal Competitive Contribution (MCC) no agent has a motive to break an edge. Similarly by results in Section 3 *4.2*, in the online Game of Maximal Marginal Contribution (MMC) no agent has a motive to break an edge. Only in the online Game of Maximal Altruistic Contribution, may an agent have a motive to break an edge. ■

Theorem 2: In an online Game of Maximal Marginal Contribution any connected graph *G* is stable. Moreover, with a game having the strengthened condition $v_2 > 2v_1$, any stable graph is connected.

Proof: In an online Game of Maximal Marginal Contribution, if the graph *G* is connected an agent *i* which is not a cut vertex will never become a cut vertex no matter how many edges it makes. Thus according to Lemma 2 it will continue to have the same minimal marginal contribution, thus it has no motive to make an edge. For an agent *i* which is a cut vertex, its marginal contribution is given by (9). In the right hand side of this, the first term is constant for a connected graph also no matter how many new edges agent *i* makes the second term will also remain unchanged. Thus agent *i* has no motive to make any edge.

The second term in the right hand side of (9) is independent of any edge incident at the agent *i*, while under the given condition $v_2 > 2v_1$, the first term is maximum only when the graph *G* is connected. Thus in an online Game of Maximal Marginal Contribution a disconnected graph cannot be stable. ■

Remark 4: Let an online Game of Maximal Marginal Contribution be started from a completely disconnected graph, and let every agent be allowed to make as many edges as it desires. Then the agent who gets the first move to make edges will make edges with all the other agents. This will make a star, which is also a tree, with the first agent at the center. Then there is no motive for any other player to make or break an edge. ■

Theorem 3: In an online Game of Maximal Competitive Contribution any complete graph is stable. Moreover, with a game having $v_2 > 2v_1$, any stable graph is complete.

Proof: In an online Game of Maximal Competitive Contribution a complete graph is stable since there is no more edge to make and there is no motive to break an edge [3] and condition *(b)* of Game of Maximal Competitive Contribution.

It is established in Lemma 1 that the competitive contribution of an agent is the same as the Positional Advantage of an agent. Whenever an edge is added in a graph, the PAs of both of its end vertices increase under the given condition [3]. Thus there is always a motive to make an edge whenever it is possible. ■

Theorem 4: In an online Game of Maximal Altruistic Contribution any tree is a stable graph. Moreover, with a game having $v_2 > 2v_1$, if a graph G is stable then it is a tree.

Proof: The altruistic contribution of an agent i is given by (11)

$$a_G(i) = \sum_{j\in G\backslash i} \varphi_G(j) - \sum_{j\in G\backslash i} \varphi_{G\backslash i}(j) \tag{15}$$

or

$$a_G(i) = \sum_{j\in G} \varphi_G(j) - \varphi_G(i) - \sum_{j\in G\backslash i} \varphi_{G\backslash i}(j) \tag{16}$$

By the axioms of the game in Definition 1 and using the known facts about the PA in (3), inherited from the Shapley value [19], all the connected graphs having the same number of agents have the same value of $\sum_{j\in G} \varphi_G(j) = v_{|G|}$. Since tree is a connected graph, formation of a new edge by any agent i does not change the value of the first term at the right hand side of (16). Moreover, for each new edge agent i makes, its PA increases or remains constant [3]. Moreover the last term at the right hand side of (16) is independent of agent i. Thus the agent i have no motive to make an edge.

Since a tree is minimally connected graph, all of its edges are cut edges and all the agents are reachable from any other vertex in the tree. Thus, breakage of any edge by the agent i, incident at it will not increase the first term in the right hand side of (15) [3]. Moreover the second term at the right hand side of (15) is independent of any edge incident at the agent i. The altruistic contribution $a_G(i)$ of the agent i thus reduces or remains constant upon the breakage of any edge incident at it. Thus agents have no motive to break any of the edge incidents at them.

The set of simple graphs can be partitioned into three classes: disconnected graphs, minimally connected graphs and non-minimally connected graphs. It is to be established that under given condition $v_2 > 2v_1$, G is neither disconnected nor it is non-minimally connected.

If G is disconnected then there always exist at least two agents i and j which are not reachable from each other. From (3) and (15), under the given condition, making of the edge fulfills the condition *(a)* of Game of Maximal Altruistic Contribution. A disconnected graph is thus unstable. If G is non-minimally connected then there must exist an edge $e = \{i, j\}$ such that G remains connected even after its removal. Removal of edge $e = \{i, j\}$ by the agent i thus does not change the first term in the right hand side of (16), moreover the last term in the right hand side of (16) is independent of any edge incident at i, and under given condition, $\varphi_G(i)$ decreases upon removal of edge e [3]. A non-minimal connected graph is also unstable. ■

Remark 5: It follows from the results established in this section that if $v_2 > 2v_1$, then in any of the three online sequential decision games defined in this section a stable graph is always connected. ■

5. Simulation Examples of Online Sequential Decision Games

Simulation results for the three sequential decision games are presented here. In these simulations the games are started from an initial graph and the agents are free to make allowed moves as in Definition 9. One agent is randomly selected to makes moves at each time, and can make or break as many edges as it desires to improve its contribution objective function. The method established in [8] for fast computation of Shapley value is used in the simulation.

The simulations were run until one of the stable graphs is reached. The simulation results support the theory developed in Section 4. These results show that any connected graph is stable in Game of Maximal Marginal Contribution (MMC), only a complete graph is stable in Game of Maximal Competitive Contribution (MCC), and any tree is stable in Game of Maximal Altruistic Contribution (MAC). Some of the simulation results are presented in the following Figures 2-4.

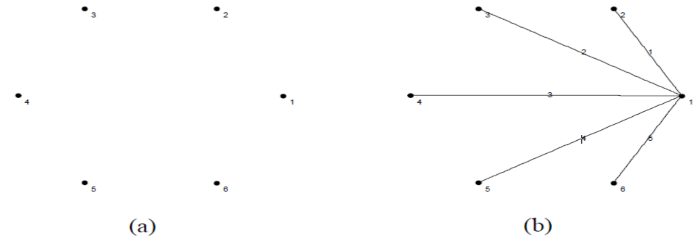

Fig. 2. Evolution of graph in MMC when agents are allowed to make or break as many edges as desired. (a) Initial, completely disconnected graph. (b) Stable Connected Graph (a tree) results after one move.

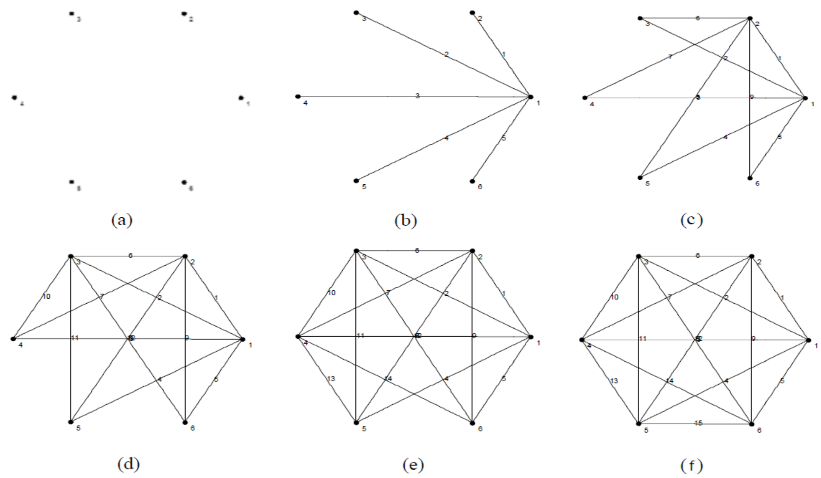

Fig. 3.Evolution of graph in MCC when agents are allowed to make or break as many edges as desired. (a) Initial, completely disconnected graph. (b)-(e) Transition states on sequential moves of randomly selected agents. (f) Stable, Complete Graph.

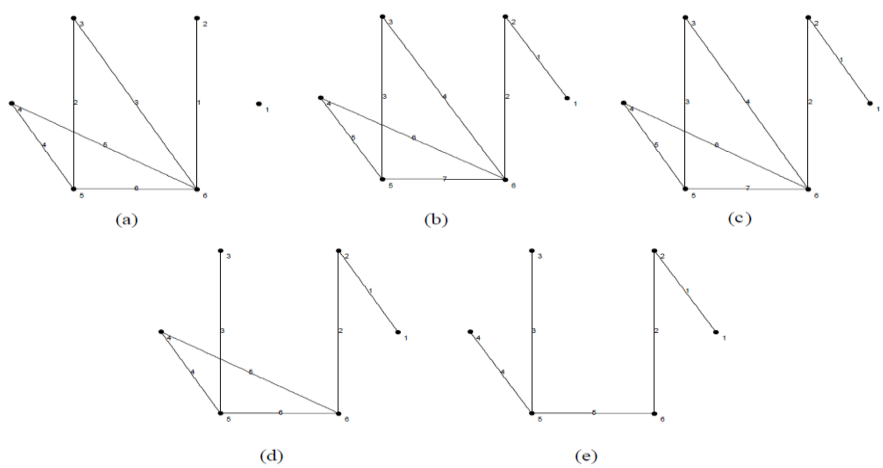

Fig. 4. Evolution of graph in MAC when agents are allowed to make or break as many edges as desired. (a) Initial, random graph. (b)-(d) Transition states on sequential moves of randomly selected agents. (e) Stable graph, a Tree.

6. Conclusion

A Graphical Coalitional Game that assigns a value to each individual in a team, represented by a communication graph, based on its connectivity is presented in this paper. The game defines the notion of Positional Advantage of agents in a coalition from a connectivity aspect. The components of the contribution made by an agent to team efforts are defined and their dependence on the connectivity in teams is established. The setup of GCG and sequential decision games provides an insight to the internal structure of coalitions, provides a guideline for the formation of coalitions, and serves to examine the competitiveness and altruism aspects of the coalition.

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